selected a point b that lay to the left of a on the x-axis as one that lay to the right.

In the event the limit L exists as h tends to 0, in line with the discussion above, we would interpret the number L to be the slope of the tangent line to the curve at the point $(a, f(a))$. Analytically, this means the tangent to the curve at $(a, f(a))$ is given by

$$g(x) = f'(a)(x-a) + f(a).$$

The reader should understand this is essentially a definition. The notion of tangent is a geometric notion that provides motivation for the definition of the derivative. However, the limit as defined above, which is analytic in nature, does not necessarily have to produce a tangent to the curve at the point $(a, f(a))$ in the geometric sense. For one thing, prior to defining the derivative, we don't have tangents to most types of curves. In such a case, where no geometric tangent is available, how are we to decide that the straight line passing through $(a, f(a))$ and having slope $f'(a)$ really is the correct tangent line? We cannot decide, so when we assert that it is, we are essentially making a definition. What we can do is check that the analytic specification for a tangent line agrees with, or has the same properties as, the geometric tangent line in cases for which the latter exists. This type of check will be illustrated in Example 5. □

We have stressed that whenever a new limit definition arises, a number of natural questions arise. For example, is the limit unique, how does it behave with respect to numerical operations, and so on. Since these results are essential to the remaining development, we state two summary theorems.

Theorem 4.1.1. *Let f be defined in a neighborhood about a and differentiable at a. Then $f'(a)$ is unique.*

Proof. We offer a sketch; details are left to Exercise 20.

Assume for the sake of argument that $f'(a)$ is not unique and both L and M satisfy the definition. Then set $\epsilon = \frac{|L-M|}{3}$. A contradiction is now easily obtained. □

Discussion. Note the similarities in the reasonings here and those of Theorems 0.4.1, 1.2.1, 2.1.1, and 2.3.1(a). □

Theorem 4.1.2. *Let f and g be defined in a neighborhood about a and differentiable at a. Then*

(i) *for any $c \in \mathbf{R}$, $(cf)'$ exists and $(cf)' = cf'$;*

(ii) *$(f+g)'(a)$ exists and $(f+g)'(a) = f'(a) + g'(a)$;*

(iii) *$(fg)'(a)$ exists and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$;*

(iv) *if $g(a) \neq 0$, $\left(\frac{f}{g}\right)'(a)$ exists and $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$.*

Proof. We prove (iii) and leave the rest to Exercises 21–24. Thus, assume f